Derivation of Transport Equation in Time-Dependent Projection Operator Method

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Abstract

We develop a formalism to carry out coarse-grainings by using a time-dependent projection operator in Heisenberg picture. A systematic perturbative expansion with respect to the interaction part of the Hamiltonian is given and the Langevin-type equation without a time-convolution integral term is obtained. This method is applied to a quantum field theoretical model and a coupled transport equations is derived.

I. INTRODUCTION

The description of the nonequilibrium process in quantum field theory is one of the important topics and have been actively studied. Then, it is often important to carry out reduction or coarse-grainings of the irrelevant degree of freedom. In classical system, it is widely believed that a system of macroscopic size composed of many microscopic variables exhibits rather simple macroscopic behavior described in terms of only a few macroscopic variables. Therefore, it may be expected that such a reduction of the irrelevant information can be applied to a quantum system and help the analysis of the nonequilibrium process. The projection operator method is one of the famous methods to carry out coarse-grainings systematically [1]– [3]. In this method, after the elimination of the irrelevant information by means of a projection operator, we obtain a kind of master equation in Schrödinger picture or Langevin-type equation in Heisenberg picture. Famous examples are the Mori and Nakajima-Zwanzig equations [1]. Recently, the unification and the generalization of the two treatments were achieved [2] [3].

In the derivation of kinetic equations from the microscopic point of view, we usually assume that the relevant part of the system is interacting with the irrelevant subsystem which is regarded as a heat reservoir in thermal equilibrium [4]. However, in the general nonequilibrium system, we can expect that the irrelevant subsystem have some time-dependence. Furthermore, it is often convenient to have a description where each of two or more coupled systems is considered to be on an equal footing. Especially in quantum field theory, both relevant and irrelevant parts of the system may be composed of infinite degrees of

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freedom respectively, and therefore, it is not clear whether we can regard one subsystem as a reservoir to the other or not. To realize these situation, it is convenient to introduce the time-dependent projection operator.

The time-dependent projection operator is first applied by Robertson [5]. He tried to project the complex behavior of the relevant density matrix on a local equilibrium density matrix and derived a master equation. Ochiai attempted to achieve a description of the kinetic stage by using the same projection [6]. Robertson's time-dependent projection operator was improved by Kawasaki et al. [7]. Willis et al. considered the case of the coupled systems and introduced another time-dependent projection operator [8]. Similar projection was implemented also by Grabert et al. [9]. Shibata et al. derived a systematic perturbative expansion formula of a master equation without time-convolution integral terms [10].

All the above works were discussed in Schrödinger picture, and therefore, the master equations were derived. On the other hand, the Langevin-type equation is obtained when we apply the projection operator method in Heisenberg picture. As long as we know, the formulation in Heisenberg picture with the time-dependent projection operator has not been discussed. In this paper, we will develop a formalism in Heisenberg picture and derive the Langevin-type equation. It is known that both the equations with and without a time-convolution integral term can be derived in the projection operator method [2] [3]. In this study, we will discuss the equation without a time-convolution integral term.

This paper is organized as follows. In Sec. II, the projection operator method with a time-dependent projection operator is derived. In Sec. III, we apply our formalism to a quantum field theoretical model, which is composed of two bosons. We assume the local equilibrium and derive the coupled transport equations. Summary and conclusions are given in Sec. IV.

II. TIME-DEPENDENT PROJECTION OPERATOR METHOD

Our starting point is the Heisenberg equation of motion,

$$\frac{d}{dt}O(t) = i[H, O(t)] \tag{1}$$

$$= iLO(t) \tag{2}$$

$$\longrightarrow O(t) = e^{iL(t-t_0)}O(t_0), \tag{3}$$

where L is the Liouville operator and t_0 is the time at which we prepare an initial state. The Heisenberg equation contains complete information of the time-evolution of the operator, but in general, it is difficult to solve exactly when there are interactions. Therefore it is necessary to carry out the reduction or the course-grainings of the irrelevant information. For this purpose, we introduce the time-dependent projection operators $P(t)$, and $Q(t)$ which is defined as

$$Q(t) = 1 - P(t). \tag{4}$$

The projection operator $P(t)$ helps us to project any operator onto the P-space which consists of the relevant degrees of freedom. From Eq. (3), one can see that the time-dependence of the operators is determined by $e^{iL(t-t_0)}$. This yields the equation

$$\begin{aligned}\frac{d}{dt}e^{iL(t-t_0)} &= e^{iL(t-t_0)}iL \\ &= e^{iL(t-t_0)}(P(t)+Q(t))iL. \end{aligned} \tag{5}$$

From this equation, we can derive the following two equations:

$$\frac{d}{dt}e^{iL(t-t_0)}P(t) = e^{iL(t-t_0)}(P(t)+Q(t))iLP(t) + e^{iL(t-t_0)}\dot{P}(t), \tag{6}$$

$$\frac{d}{dt}e^{iL(t-t_0)}Q(t) = e^{iL(t-t_0)}(P(t)+Q(t))iLQ(t) + e^{iL(t-t_0)}\dot{Q}(t), \tag{7}$$

where $\dot{P}(t) = dP(t)/dt$ and $\dot{Q}(t) = dQ(t)/dt$. Eq. (7) can be solved for $e^{iL(t-t_0)}Q$:

$$e^{iL(t-t_0)}Q(t) = Q(t_0)e_{\rightarrow}^{i\int_{t_0}^{t} dsLQ(s)} + \int_{t_0}^{t} dse^{iL(s-t_0)}(\dot{Q}(s)+P(s)iLQ(s))e_{\rightarrow}^{i\int_{s}^{t} d\tau LQ(\tau)} \tag{8}$$

$$\begin{aligned} &= Q(t_0)e_{\rightarrow}^{i\int_{t_0}^{t} dsLQ(s)} + e^{iL(t-t_0)}(P(t)+Q(t))\Sigma(t,t_0) \\ &= \{Q(t_0)e_{\rightarrow}^{i\int_{t_0}^{t} dsLQ(s)} + e^{iL(t-t_0)}P(t)\Sigma(t,t_0)\}\frac{1}{1-\Sigma(t,t_0)}, \end{aligned} \tag{9}$$

where

$$\Sigma(t,t_0) = \int_{t_0}^{t} dse^{-iL(t-s)}\{\dot{Q}(s)+P(s)iLQ(s)\}e_{\rightarrow}^{i\int_{s}^{t} d\tau LQ(\tau)}. \tag{10}$$

Here, the time ordered operator $e_{\rightarrow}^{i\int_{t_0}^{t} dsLQ(s)}$ is defined as

$$e_{\rightarrow}^{i\int_{t_0}^{t} dsLQ(s)} = 1 + \sum_{n=1}^{\infty} i^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n LQ(t_n)LQ(t_{n-1})\cdots LQ(t_1). \tag{11}$$

Note that there is a term including $\dot{Q}(s)$ which is not observed in the the time-independent projection operator method [3].

Substituting Eq. (9) into Eq. (5) and operating with $O(t_0)$ from the right, we obtain

$$\begin{aligned}\frac{d}{dt}O(t) = {} & e^{iL(t-t_0)}P(t)iLO(t_0) \\ & +e^{iL(t-t_0)}P(t)\Sigma(t,t_0)\frac{1}{1-\Sigma(t,t_0)}iLO(t_0) \\ & +Q(t_0)e_{\rightarrow}^{i\int_{t_0}^{t} dsLQ(s)}\frac{1}{1-\Sigma(t,t_0)}iL(t,t_0)O(t_0) \end{aligned} \tag{12}$$

This equation has no time-convolution integral. This point is demonstrated at the end of this section. We call this the time-convolutionless (TCL) equation. When the time-dependence of the projection operator is ignored, this equation agrees with the equation(2·13) in Ref. [3]. When we substitute Eq. (8) into Eq. (5), we obtain the equation with a time-convolution integral term, which is called the time-convolution(TC) equation [2] [3]. However, we will discuss only about the TCL equation in this paper.

The TCL equation is still exactly equivalent to the Heisenberg equation of motion. In usual, we carry out the perturbative expansion of the interaction Hamiltonian and take only the lowest order terms which are sufficient to describe the dissipation effect. However the above TCL equation is not convenient to carry out the perturbative expansion. Next our task is to rewrite the TCL equation. For this purpose, we restrict the nature of the projection operator. In the time-independent projection operator, $P^2 = P$ is satisfied. On the other hand, in the time-dependent case, the order of the operation of the projection operators is important. In the previous works, the condition $P(t_1)P(t_2) = P(t_1)$ was employed [5]– [10]. However, in this paper, we assume the condition

$$P(t_1)P(t_2) = P(t_2), \tag{13}$$

because the projection operator which we will use in the later section satisfies this condition. From this condition, we can derive several relations:

$$Q(t_1)P(t_2) = 0, \tag{14}$$
$$P(t_1)Q(t_2) = P(t_1) - P(t_2), \tag{15}$$
$$Q(t_1)Q(t_2) = Q(t_1). \tag{16}$$

The total Hamiltonian of the system can be divided into two parts

$$H = H_0(t, t_0) + H_I(t, t_0), \tag{17}$$

where $H_0(t, t_0)$ and $H_I(t, t_0)$ are the nonperturbative part and the perturbative part of the Hamiltonian, respectively. In the general nonequilibrium process, we can consider the case where the mass of a particle changes with time, and therefore, $H_0(t, t_0)$ becomes time-dependent. The corresponding Liouville operators are defined as

$$L_0(t, t_0)O = [H_0(t, t_0), O], \quad L_I(t, t_0)O = [H_I(t, t_0), O]. \tag{18}$$

Now, we assume another condition

$$Q(t_2)L_0(t_1, t_0)P(t_1) = 0. \tag{19}$$

Because of this condition, the form of the nonperturbative Hamiltonian affects that of the projection operator, and vice versa. This condition means that the nonperturbative Hamiltonian does not drive the operator once projected onto the P-space into the Q-space which is orthogonal to the P-space.

With the above properties of the projection operators, the function $\Sigma(t, t_0)$ can be expressed as

$$\begin{aligned}
\Sigma(t, t_0) &= \int_{t_0}^{t} ds e^{-iL(t-s)}\{\dot{Q}(s) + P(s)iLQ(s)\}e_{\rightarrow}^{i\int_s^t d\tau LQ(\tau)} \\
&= Q(t) - e^{-iL(t-t_0)}Q(t_0)e_{\rightarrow}^{i\int_{t_0}^t d\tau LQ(\tau)} \\
&= Q(t) - U_0^{-1}(t, t_0)\mathcal{C}(t, t_0)Q(t_0)\mathcal{D}(t, t_0)e_{\rightarrow}^{i\int_{t_0}^t ds L_0(s,t_0)Q(s)}.
\end{aligned} \tag{20}$$

Here, the operators $\mathcal{C}(t,t_0)$ and $\mathcal{D}(t,t_0)$ are expressed as

$$\begin{aligned}\mathcal{C}(t,t_0) &= U_0(t,t_0)e^{-iL(t-t_0)} \\ &= 1+\sum_{n=1}^{\infty}(-i)^n\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\cdots\int_{t_0}^{t_{n-1}}dt_n\breve{L}_I(t_1,t_0)\breve{L}_I(t_2,t_0)\cdots\breve{L}_I(t_n,t_0), \end{aligned} \tag{21}$$

$$\begin{aligned}\mathcal{D}(t,t_0) &= e_{\rightarrow}^{i\int_{t_0}^{t}dsLQ(s)}(U_0^Q)^{-1}(t,t_0) \\ &= 1+\sum_{n=1}^{\infty}i^n\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\cdots\int_{t_0}^{t_{n-1}}dt_n\breve{L}_I^Q(t_1,t_0)\breve{L}_I^Q(t_2,t_0)\cdots\breve{L}_I^Q(t_n,t_0), \end{aligned} \tag{22}$$

where

$$\breve{L}_I(t,t_0) = U_0(t,t_0)L_IU_0^{-1}(t,t_0), \tag{23}$$
$$\breve{L}_I^Q(t,t_0) = U_0^Q(t,t_0)L_IQ(U_0^Q)^{-1}(t,t_0), \tag{24}$$
$$U_0(t,t_0) = e_{\rightarrow}^{i\int_{t_0}^{t}dsL_0(s,t_0)}, \tag{25}$$
$$U_0^Q(t,t_0) = e_{\rightarrow}^{i\int_{t_0}^{t}dsL_0(s,t_0)Q(s)}. \tag{26}$$

It is remarkable that the term including the time-derivative of the projection operator disappears. The operator $\mathcal{C}(t,t_0)$ $[\mathcal{D}(t,t_0)]$ is a time [an anti-time] ordered function of Liouville operators. The details of these expressions are given in Appendix A. Then, we have

$$P(t)\Sigma(t,t_0)\frac{1}{1-\Sigma(t,t_0)} = -P(t)U_0^{-1}(t,t_0)\mathcal{C}(t,t_0)Q(t)\frac{1}{1+(\mathcal{C}(t,t_0)-1)Q(t)}U_0(t,t_0). \tag{27}$$

To derive this expression, we have used the mathematical induction [3]. The detailed derivation appears in Appendix B. Substituting the above result into Eq. (12), we obtain

$$\begin{aligned}\frac{d}{dt}O(t) &= e^{iL(t-t_0)}P(t)iLO(t_0) \\ &\quad +e^{iL(t-t_0)}P(t)U_0^{-1}(t,t_0)\mathcal{C}(t,t_0)Q(t)\frac{1}{1-(\mathcal{C}(t,t_0)-1)Q(t)}U_0(t,t_0)iLO(t_0) \\ &\quad +Q(t_0)e_{\rightarrow}^{i\int_{t_0}^{t}dsLQ(s)}\frac{1}{1-\Sigma(t,t_0)}iLO(t_0). \end{aligned} \tag{28}$$

This form of the TCL equation is convenient to carry out the perturbative expansion.

When we expand $P\Sigma(t,t_0)/(1-\Sigma(t,t_0))$ up to first order in the interaction H_I, we have

$$\begin{aligned}\frac{d}{dt}O(t) &= e^{iL(t-t_0)}P(t)U_0^{-1}(t,t_0)P(t)U_0(t,t_0)iLO(t_0) \\ &\quad +e^{iL(t-t_0)}P(t)U_0^{-1}(t,t_0)P(t)\int_{t_0}^{t}dsU_0(s,t_0)iL_IU_0^{-1}(s,t_0)Q(t)U_0(t,t_0)iLO(t_0) \\ &\quad +Q(t_0)e_{\rightarrow}^{i\int_{t_0}^{t}dsLQ(s)}\frac{1}{1-\Sigma(t,t_0)}iLO(t_0). \end{aligned} \tag{29}$$

It is easily seen that Eq. (29) does not contain a time-convolution integral. If this did contain a time-convolution integral, the form of the full time-evolution operator $e^{iL(t-t_0)}$, which operates from the left in the second term on the r.h.s. of the equation, must be $e^{iL(t-s)}$, where s is an integral variable [2] [3]. Therefore, this equation is called the TCL equation.

When we ignore the time-dependence of the projection operator, this equation agrees with Eq. (2·31) in Ref. [3]. It should be noted that not $Q(t)$ but $Q(t_0)$ operates from the left in the third term on the r.h.s. of the equation. Then, this term represents the effect of the initial correlation of the initial density matrix. This point becomes more clear in the next section. In the usual projection operator method, such a term is interpreted as the fluctuation force. Therefore, the third term can be regarded as the fluctuation force also in this time-dependent projection operator method.

III. COUPLED TRANSPORT EQUATIONS

Now, we apply the time-dependent projection operator method to a quantum field theoretical model and derive a transport equation. We consider the following Hamiltonian with two boson fields

$$H = H_\sigma + H_\pi + H_I, \tag{30}$$

where

$$H_\sigma = \int d^3\mathbf{x}\frac{1}{2}\{\Phi^2(x) + (\nabla\phi(x))^2 + m_\sigma^2\phi^2(x)\}, \tag{31}$$

$$H_\pi = \int d^3\mathbf{x}\frac{1}{2}\{\Pi^2(x) + (\nabla\pi(x))^2 + m_\pi^2\pi^2(x)\}, \tag{32}$$

$$H_I = \int d^3\mathbf{x} g\pi^2(x)\phi(x). \tag{33}$$

Here, $\Phi(x)$ and $\Pi(x)$ are the conjugate fields of $\phi(x)$ and $\pi(x)$, respectively. We call particle represented by the $\phi(x)$ field the σ boson, the $\pi(x)$ field the π boson. The nonperturbative and the interaction Hamiltonian are given by $H_0 = H_\sigma + H_\pi$ and H_I, respectively. Here, we ignore the time-dependence of the nonperturbative Hamiltonian for simplicity.

These four fields are expanded as

$$\phi(\mathbf{x}, t_0) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}^\sigma}}(a_{\mathbf{k}}(t_0)e^{i\mathbf{kx}} + a_{\mathbf{k}}^\dagger(t_0)e^{-i\mathbf{kx}}), \tag{34}$$

$$\Phi(\mathbf{x}, t_0) = -i\sum_{\mathbf{k}} \sqrt{\frac{\omega_{\mathbf{k}}^\sigma}{2V}}(a_{\mathbf{k}}(t_0)e^{i\mathbf{kx}} - a_{\mathbf{k}}^\dagger(t_0)e^{-i\mathbf{kx}}), \tag{35}$$

$$\pi(\mathbf{x}, t_0) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}^\pi}}(b_{\mathbf{k}}(t_0)e^{i\mathbf{kx}} + b_{\mathbf{k}}^\dagger(t_0)e^{-i\mathbf{kx}}), \tag{36}$$

$$\Pi(\mathbf{x}, t_0) = -i\sum_{\mathbf{k}} \sqrt{\frac{\omega_{\mathbf{k}}^\pi}{2V}}(b_{\mathbf{k}}(t_0)e^{i\mathbf{kx}} - b_{\mathbf{k}}^\dagger(t_0)e^{-i\mathbf{kx}}), \tag{37}$$

where $\omega_{\mathbf{k}}^\sigma = \sqrt{\mathbf{k}^2 + m_\sigma^2}$ and $\omega_{\mathbf{k}}^\pi = \sqrt{\mathbf{k}^2 + m_\pi^2}$. Here, V and t_0 are the volume of the total system and the initial time at which we prepare an initial state. We take the limit $V \longrightarrow \infty$

at the end of the calculation. The creation and annihilation operators of the σ and π particle are subject to the following commutation relations:

$$[a_{\mathbf{k}}(t_0), a^\dagger_{\mathbf{k}'}(t_0)] = [b_{\mathbf{k}}(t_0), b^\dagger_{\mathbf{k}'}(t_0)] = \delta^{(3)}_{\mathbf{k},\mathbf{k}'}. \tag{38}$$

Here, [ ] is the commutation relation. Any other commutation relations become zeros.

For simplicity, we consider the case where the initial density matrix is given by the direct product of the σ boson and the π boson density matrices

$$\rho_0 = \rho_{0\sigma} \otimes \rho_{0\pi}. \tag{39}$$

This means that there is no initial correlation between the σ boson and the π boson.

First, we calculate the transport equation of the σ boson. In this case, the σ boson is regarded as the system and the π boson as the environment, respectively. To integrate out the environment degree of freedom, we define the time-dependent projection operator as

$$P(t)O = \mathrm{Tr}_E[\rho_E(t)O], \tag{40}$$

where

$$\rho_E(t) = e^{-\beta(t)H_\pi}/Z_\pi(t), \tag{41}$$

$$Z_\pi(t) = \mathrm{Tr}[e^{-\beta(t)H_\pi}]. \tag{42}$$

Here, $\rho_E(t)$ is the local equilibrium density matrix. This projection operator satisfies the conditions (13) and (19). This choice of $\rho_E(t)$ comes from our implicit assumption that the time-evolution of the π boson is well-approximated by the local equilibrium density matrix with the time-dependent temperature $\beta^{-1}(t)$. It is possible to calculate the transport equation without assuming the local equilibrium in the time-evolution of the system. However, in such a case, it is necessary to calculate a large number of correlation functions. To avoid this difficulty, we assume the local equilibrium in this paper. Now, we employ the following initial condition:

$$\rho_E(t_0) = \rho_{0\pi}. \tag{43}$$

This condition is needed to eliminate the contribution from the third term on the r.h.s. of Eq. (29). This becomes clear in the next paragraph. The time-dependence of the temperature is determined later.

Substituting $O(t_0) = a^\dagger_{\mathbf{k}}(t_0)a_{\mathbf{k}}(t_0)$ into Eq. (12) and using the definition (40), we obtain

$$\begin{aligned}
\frac{d}{dt}a^\dagger_{\mathbf{k}}(t)a_{\mathbf{k}}(t) = {} & \frac{ig}{\sqrt{2V\omega^\sigma_{\mathbf{k}}}}\langle\rho_{\mathbf{k}}(t,t_0)\rangle_t(a_{\mathbf{k}}(t) - a^\dagger_{\mathbf{k}}(t)) \\
& + \frac{g^2}{4V\omega^\sigma_{\mathbf{k}}}\int_{t_0}^{t} ds\{\langle[\rho_{-\mathbf{k}}(s,t_0), \rho_{\mathbf{k}}(t,t_0)]_+\rangle_t e^{i\omega^\sigma_{\mathbf{k}}(s-t)} + \langle[\rho_{\mathbf{k}}(s,t_0), \rho_{-\mathbf{k}}(t,t_0)]_+\rangle_t e^{-i\omega^\sigma_{\mathbf{k}}(s-t)} \\
& -2\langle\rho_{-\mathbf{k}}(s,t_0)\rangle_t\langle\rho_{\mathbf{k}}(t,t_0)\rangle_t e^{i\omega^\sigma_{\mathbf{k}}(s-t)} - 2\langle\rho_{-\mathbf{k}}(s,t_0)\rangle_t\langle\rho_{\mathbf{k}}(t,t_0)\rangle_t e^{-i\omega^\sigma_{\mathbf{k}}(s-t)}\} \\
& - \frac{g^2}{4V\omega^\sigma_{\mathbf{k}}}\int_{t_0}^{t} ds\{\langle[\rho_{-\mathbf{k}}(s,t_0), \rho_{\mathbf{k}}(t,t_0)]\rangle_t[a_{\mathbf{k}}(t), a_{-\mathbf{k}}(t)e^{-i\omega^\sigma_{\mathbf{k}}(s-t)} + a^\dagger_{\mathbf{k}}(t)e^{i\omega^\sigma_{\mathbf{k}}(s-t)}]_+ \\
& -\langle[\rho_{\mathbf{k}}(s,t_0), \rho_{-\mathbf{k}}(t,t_0)]\rangle_t[a^\dagger_{\mathbf{k}}(t), a_{\mathbf{k}}(t)e^{-i\omega^\sigma_{\mathbf{k}}(s-t)} + a^\dagger_{-\mathbf{k}}(t)e^{i\omega^\sigma_{\mathbf{k}}(s-t)}]_+\} \\
& + flu,
\end{aligned} \tag{44}$$

where

$$\langle\rho_{\mathbf{k}}(t,t_0)\rangle_t = \int d^3\mathbf{x} e^{i\mathbf{k}\mathbf{x}} \mathrm{Tr}[\rho_E(t)\pi^2(\mathbf{x},t;t_0)], \tag{45}$$

$$\langle[\rho_{\mathbf{k_1}}(s,t_0),\rho_{\mathbf{k_2}}(t,t_0)]\rangle_t = \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 e^{i\mathbf{k_1}\mathbf{x_1}} e^{i\mathbf{k_2}\mathbf{x_2}} \mathrm{Tr}[\rho_E(t)[\pi^2(\mathbf{x}_1,s;t_0),\pi^2(\mathbf{x}_2,t;t_0)]]. \tag{46}$$

$$\langle[\rho_{\mathbf{k_1}}(s,t_0),\rho_{\mathbf{k_2}}(t,t_0)]_+\rangle_t = \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 e^{i\mathbf{k_1}\mathbf{x_1}} e^{i\mathbf{k_2}\mathbf{x_2}} \mathrm{Tr}[\rho_E(t)[\pi^2(\mathbf{x}_1,s;t_0),\pi^2(\mathbf{x}_2,t;t_0)]_+]. \tag{47}$$

Here, $[\ ]_+$ is the anti-commutation relation. The last term flu on the r.h.s. of the equation comes from the third term on the r.h.s. of Eq. (29). When we calculate the expectation value by the initial density matrix, this term vanishes because of the definition of flu and the condition (43). When we consider the initial density matrix with an initial correlation, we cannot employ the condition (39), and the third term has a finite value. In short, the third term represents the effect of the initial correlation. We can thus regard the third term flu as the fluctuation force term as in the usual projection operator method. In this calculation, we do not consider the initial correlation, and therefore, we ignore this term.

Before making the calculation advance, we will discuss the structure of this equation. This equation has the strange terms which are proportional to $a^\dagger_{\mathbf{k}}(t)$, $a_{\mathbf{k}}(t)$, $a^\dagger_{\mathbf{k}}(t)a^\dagger_{-\mathbf{k}}(t)$ and $a_{\mathbf{k}}(t)a_{-\mathbf{k}}(t)$. Such terms are not seen in the usual Boltzmann equation which is constituted of the terms proportional to $a^\dagger_{\mathbf{k}}(t)a_{\mathbf{k}}(t)$. The terms proportional to $a^\dagger_{\mathbf{k}}(t)$ and $a_{\mathbf{k}}(t)$ survive when we consider the case where there is the condensate of the σ boson. Here, we do not consider the case of the condensate, and therefore, we ignore such terms. The existence of the terms proportional to $a^\dagger_{\mathbf{k}}(t)a^\dagger_{-\mathbf{k}}(t)$ and $a_{\mathbf{k}}(t)a_{-\mathbf{k}}(t)$ has already been pointed out in Ref. [11]. They have pointed out that the existence of such terms is homologous to the parametric amplifier, and $a^\dagger_{\mathbf{k}}(t)a^\dagger_{-\mathbf{k}}(t)$, $a_{\mathbf{k}}(t)a_{-\mathbf{k}}(t)$ and $a^\dagger_{\mathbf{k}}(t)a_{\mathbf{k}}(t)$ form the SU(1,1) symmetry group. In this calculation, we drop such terms for simplicity.

To calculate the correlation functions (45), (46) and (47), it is convenient to use the technique in thermo field dynamics(TFD). In TFD, the statistical average is expressed by a kind of a vacuum expectation value. As an example, we consider the statistical average of the system with the free Hamiltonian of a boson system $H = \sum \omega_{\mathbf{k}} d^\dagger_{\mathbf{k}} d_{\mathbf{k}}$. First, we introduce the transformation which introduces the new creation-annihilation operators $D_{\mathbf{k}}, D^\dagger_{\mathbf{k}}$ and $\tilde{D}_{\mathbf{k}}, \tilde{D}^\dagger_{\mathbf{k}}$:

$$d^\dagger_{\mathbf{k}} = \cosh\theta_{\mathbf{k}} D^\dagger_{\mathbf{k}} + \sinh\theta_{\mathbf{k}} \tilde{D}_{\mathbf{k}}, \tag{48}$$

$$d_{\mathbf{k}} = \cosh\theta_{\mathbf{k}} D_{\mathbf{k}} + \sinh\theta_{\mathbf{k}} \tilde{D}^\dagger_{\mathbf{k}}, \tag{49}$$

where

$$\sinh^2\theta_{\mathbf{k}} = n_{\mathbf{k}}, \quad \cosh^2\theta_{\mathbf{k}} = 1 + n_{\mathbf{k}}. \tag{50}$$

Here, $n_{\mathbf{k}}$ is a Bose distribution function. The above operators satisfy the following commutation relation

$$[D_{\mathbf{k}}, D^\dagger_{\mathbf{k}'}] = [\tilde{D}_{\mathbf{k}}, \tilde{D}^\dagger_{\mathbf{k}'}] = \delta^{(3)}_{\mathbf{k},\mathbf{k}'}. \tag{51}$$

Any other commutation relations become zeros. Now, we can define a thermal vacuum $|\theta_D\rangle$ as

$$D_{\mathbf{k}}|\theta_D\rangle = \tilde{D}_{\mathbf{k}}|\theta_D\rangle = 0. \tag{52}$$

We thus express the statistical average by the vacuum expectation value:

$$\mathrm{Tr}[\rho_{th} O] = \langle\theta_D|O|\theta_D\rangle, \tag{53}$$

where

$$\rho_{th} = e^{-\beta H}/Z, \tag{54}$$
$$Z = \mathrm{Tr}\ e^{-\beta H}. \tag{55}$$

Now, we return to the calculation of the correlation functions with the local equilibrium density matrix. In the case of the time-dependent temperature, we introduce the time-dependent transformation to introduce the new creation-annihilation operators $B_{\mathbf{k}}(t), B_{\mathbf{k}}^\dagger(t)$ and $\tilde{B}_{\mathbf{k}}(t), \tilde{B}_{\mathbf{k}}^\dagger(t)$:

$$b_{\mathbf{k}}^\dagger(t_0) = \cosh\theta_{\mathbf{k}}^\pi(t) B_{\mathbf{k}}^\dagger(t) + \sinh\theta_{\mathbf{k}}^\pi(t)\tilde{B}_{\mathbf{k}}(t), \tag{56}$$
$$b_{\mathbf{k}}(t_0) = \cosh\theta_{\mathbf{k}}^\pi(t) B_{\mathbf{k}}(t) + \sinh\theta_{\mathbf{k}}^\pi(t)\tilde{B}_{\mathbf{k}}^\dagger(t), \tag{57}$$

where

$$\sinh^2\theta_{\mathbf{k}}^\pi(t) = n_{\mathbf{k}}^\pi(t), \tag{58}$$
$$\cosh^2\theta_{\mathbf{k}}^\pi(t) = 1 + n_{\mathbf{k}}^\pi(t). \tag{59}$$

Here, $n_{\mathbf{k}}^\pi(t)$ is the time-dependent distribution function of the π boson

$$\begin{aligned} n_{\mathbf{k}}^\pi(t) &= \mathrm{Tr}[\rho_E(t) b_{\mathbf{k}}^\dagger(t_0) b_{\mathbf{k}}(t_0)] \\ &= \frac{1}{e^{\beta(t)\omega_{\mathbf{k}}^\pi} - 1}. \end{aligned} \tag{60}$$

The creation-annihilation operators $B_{\mathbf{k}}(t), B_{\mathbf{k}}^\dagger(t)$ and $\tilde{B}_{\mathbf{k}}(t), \tilde{B}_{\mathbf{k}}^\dagger(t)$ satisfy

$$[B_{\mathbf{k}}(t), B_{\mathbf{k}'}^\dagger(t)] = [\tilde{B}_{\mathbf{k}}(t), \tilde{B}_{\mathbf{k}'}^\dagger(t)] = \delta_{\mathbf{k},\mathbf{k}'}^{(3)}. \tag{61}$$

Any other commutation relations vanish. Then, we can define the time-dependent vacuum $|\theta_B(t)\rangle$ as

$$B_{\mathbf{k}}(t)|\theta_B(t)\rangle = \tilde{B}_{\mathbf{k}}(t)|\theta_B(t)\rangle = 0. \tag{62}$$

We thus express the statistical average with the local equilibrium density matrix in terms of the time-dependent vacuum expectation value:

$$\mathrm{Tr}[\rho_E(t) O] = \langle\theta_B(t)|O|\theta_B(t)\rangle, \tag{63}$$

where $\rho_E(t)$ is defined in Eq. (41). Therefore, we can simplify the calculation of the correlation functions with the help of the Wick's theorem [14]. In short, we obtain

$$
\begin{aligned}
\mathrm{Tr}[\rho_E(t)\rho_{\mathbf{k}}(t,t_0)] &= \langle\theta_B(t)|\rho_{\mathbf{k}}(t,t_0)|\theta_B(t)\rangle \\
&= \int d^3\mathbf{x}e^{i\mathbf{kx}}\underbrace{\pi(\mathbf{x},t)\,\pi}_{\mathrm{B(t)}}(\mathbf{x},t) \qquad (64)\\
\mathrm{Tr}[\rho_E(t)[\rho_{\mathbf{k_1}}(s,t_0),\rho_{\mathbf{k_2}}(t,t_0)]] &= \langle\theta_B(t)|[\rho_{\mathbf{l}}(s,t_0),\rho_{\mathbf{k}}(t,t_0)]|\theta_B(t)\rangle \\
&= \int d^3\mathbf{x_1}d^3\mathbf{x_2}e^{i\mathbf{k_1x_1}}e^{i\mathbf{k_2x_2}} \\
&\quad \times 2\{(\underbrace{\pi(\mathbf{x}_1,s)\,\pi}_{\mathrm{B(t)}}(\mathbf{x}_2,t))^2-(\underbrace{\pi(\mathbf{x}_2,t)\,\pi}_{\mathrm{B(t)}}(\mathbf{x}_1,s))^2\}, \qquad (65)
\end{aligned}
$$

where

$$
\begin{aligned}
\underbrace{\pi(\mathbf{x}_1,t_1)\,\pi}_{\mathrm{B(t)}}(\mathbf{x}_2,t_2) = &\sum_{\mathbf{k_1},\mathbf{k_2}}\frac{1}{2V\sqrt{\omega^\pi_{\mathbf{k_1}}\omega^\pi_{\mathbf{k_2}}}}[\cosh^\pi_{\mathbf{k_1}}(t)B_{\mathbf{k_1}}(t)e^{-i\omega^\pi_{\mathbf{k_1}}(t_1-t_0)}+\sinh\theta^\pi_{-\mathbf{k_1}}(t)\tilde{B}_{-\mathbf{k_1}}(t)e^{i\omega^\pi_{\mathbf{k_1}}(t_1-t_0)}, \\
&\sinh\theta^\pi_{-\mathbf{k_2}}(t)\tilde{B}^\dagger_{-\mathbf{k_2}}(t)e^{-i\omega^\pi_{\mathbf{k_2}}(t_2-t_0)}+\cosh^\pi_{\mathbf{k_2}}(t)B^\dagger_{\mathbf{k_2}}(t)e^{i\omega^\pi_{\mathbf{k_2}}(t_2-t_0)}] \\
= &\sum_{\mathbf{k}}\frac{1}{2V\omega^\pi_{\mathbf{k}}}(\cosh^2\theta^\pi_{\mathbf{k}}(t)e^{-i\omega^\pi_{\mathbf{k}}(t_1-t_2)}+\sinh^2\theta^\pi_{\mathbf{k}}(t)e^{i\omega^\pi_{\mathbf{k}}(t_1-t_2)})e^{i\mathbf{k}(\mathbf{x_1}-\mathbf{x_2})}, \qquad (66)
\end{aligned}
$$

One who knows the nonequilibrium thermo field dynamics may notice the difference between the technique which we have shown here and that in Ref. [12]. However, both give the same result as for the trace calculation [13].

Substituting the above results and taking the expectation value with respect to the initial density matrix, we obtain the transport equation of the σ distribution function

$$
\begin{aligned}
\frac{d}{dt}n^\sigma_{\mathbf{k}} = &\frac{g^2}{2(2\pi)^3\omega^\sigma_{\mathbf{k}}}\int_{t_0}^t ds\int d^3\mathbf{q}\frac{1}{\omega^\pi_{\mathbf{q}}\omega^\pi_{\mathbf{q+k}}} \\
&\times[\{(1+n^\pi_{\mathbf{q}})(1+n^\pi_{\mathbf{q+k}})(1+n^\sigma_{\mathbf{k}})-n^\pi_{\mathbf{q}}n^\pi_{\mathbf{q+k}}n^\sigma_{\mathbf{k}}\}\cos(\omega^\pi_{\mathbf{q}}+\omega^\pi_{\mathbf{q+k}}+\omega^\sigma_{\mathbf{k}})(s-t) \\
&+\{n^\pi_{\mathbf{q}}n^\pi_{\mathbf{q+k}}(1+n^\sigma_{\mathbf{k}})-(1+n^\pi_{\mathbf{q}})(1+n^\pi_{\mathbf{q+k}})n^\sigma_{\mathbf{k}}\}\cos(\omega^\pi_{\mathbf{q}}+\omega^\pi_{\mathbf{q+k}}-\omega^\sigma_{\mathbf{k}})(s-t) \\
&+\{(1+n^\pi_{\mathbf{q}})n^\pi_{\mathbf{q+k}}(1+n^\sigma_{\mathbf{k}})-n^\pi_{\mathbf{q}}(1+n^\pi_{\mathbf{q+k}})n^\sigma_{\mathbf{k}}\}\cos(\omega^\pi_{\mathbf{q}}-\omega^\pi_{\mathbf{q+k}}+\omega^\sigma_{\mathbf{k}})(s-t) \\
&+\{n^\pi_{\mathbf{q}}(1+n^\pi_{\mathbf{q+k}})(1+n^\sigma_{\mathbf{k}})-(1+n^\pi_{\mathbf{q}})n^\pi_{\mathbf{q+k}}n^\sigma_{\mathbf{k}}\}\cos(\omega^\pi_{\mathbf{q}}-\omega^\pi_{\mathbf{q+k}}-\omega^\sigma_{\mathbf{k}})(s-t)], \qquad (67)
\end{aligned}
$$

where $n^\sigma_{\mathbf{k}}(t)=\mathrm{Tr}[\rho a^\dagger_{\mathbf{k}}(t_0)a_{\mathbf{k}}(t_0)]$. We have omitted the time-dependence of $n^\sigma_{\mathbf{k}}$ and $n^\pi_{\mathbf{k}}$ for simplicity. Each term in the brace on the r.h.s. of the equation is composed of two contributions:the gain term and the loss term. The first term describes the creation of two π bosons and one σ boson minus the annihilation of them. The second term describe the creation of one σ boson and annihilation of two π bosons minus the annihilation of one σ boson and creation of two π bosons. The third and fourth terms describe the creation of one σ boson and one π boson and annihilation of one π boson minus the annihilation of one σ boson and one π boson and creation of one π boson. This is a reasonable result which is expected from the usual Boltzmann equation [12] [15]. When we take the limit $t_0\longrightarrow-\infty$, the Dirac delta functions which preserve the energy conservation is obtained.

To solve the above transport equation, it is necessary to determine the time-dependence of the π distribution function. In this paper, we calculate it by solving the transport equation of the π boson which is derived in the same procedure as the calculation of Eq. (67). In short, we obtain the coupled transport equations of the σ and π distribution function.

Now, we regard the π boson as the system and the σ boson as the environment. The projection operator is given by

$$P(t)O = \mathrm{Tr}_E[\rho_E(t)O], \tag{68}$$

where

$$\rho_E(t) = e^{-\beta(t)H_\sigma}/Z_\phi(t), \tag{69}$$
$$Z_\phi(t) = \mathrm{Tr}[e^{-\beta(t)H_\sigma}]. \tag{70}$$

As in the case of the σ boson, the σ distribution function is included in the π transport equation. We suppose the local equilibrium in the time-evolution again, i.e.,

$$\mathrm{Tr}[\rho_E(t)a^\dagger_{\mathbf{k}}(t_0)a_{\mathbf{k}}(t_0)] = n^\sigma_{\mathbf{k}}(t), \tag{71}$$

where $n^\sigma_{\mathbf{k}}(t)$ is given by the solution of Eq. (67)

When we use the definition (69) and substitute $O(t_0) = b^\dagger(t_0)b(t_0)$ into Eq. (12), we can obtain the transport equation of the π distribution function. In this case, the transport equation has fourth order correlation functions of the π boson. We thus approximate such terms as

$$\mathrm{Tr}[\rho_0 b^\dagger_{\mathbf{k}}(t)b^\dagger_{\mathbf{l}}(t)b_{\mathbf{m}}(t)b_{\mathbf{n}}(t)] = n^\pi_{\mathbf{k}}(t)n^\pi_{\mathbf{l}}(t)\delta^{(3)}_{\mathbf{k},\mathbf{m}}\delta^{(3)}_{\mathbf{l},\mathbf{n}} + n^\pi_{\mathbf{k}}(t)n^\pi_{\mathbf{l}}(t)\delta^{(3)}_{\mathbf{k},\mathbf{n}}\delta^{(3)}_{\mathbf{l},\mathbf{m}}. \tag{72}$$

Finally, the transport equation of the π distribution function is

$$\begin{aligned}\frac{d}{dt}n^\pi_{\mathbf{k}} = \int_{t_0}^t ds \int d^3\mathbf{l}\frac{g^2}{(2\pi)^3\omega^\pi_{\mathbf{k}}\omega^\pi_{\mathbf{l}}\omega^\sigma_{\mathbf{l+k}}}&\\ \times[\{(1+n^\sigma_{\mathbf{k+l}})(1+n^\pi_{\mathbf{k}})(1+n^\pi_{\mathbf{l}}) - n^\sigma_{\mathbf{k+l}}n^\pi_{\mathbf{k}}n^\pi_{\mathbf{l}}\}\cos(\omega^\sigma_{\mathbf{l+k}}+\omega^\pi_{\mathbf{l}}+\omega^\pi_{\mathbf{k}})(s-t)&\\ +\{n^\sigma_{\mathbf{k+l}}(1+n^\pi_{\mathbf{k}})(1+n^\pi_{\mathbf{l}}) - (1+n^\sigma_{\mathbf{k+l}})n^\pi_{\mathbf{k}}n^\pi_{\mathbf{l}}\}\cos(\omega^\sigma_{\mathbf{l+k}}-\omega^\pi_{\mathbf{l}}-\omega^\pi_{\mathbf{k}})(s-t)&\\ +\{(1+n^\sigma_{\mathbf{k+l}})(1+n^\pi_{\mathbf{k}})n^\pi_{\mathbf{l}} - n^\sigma_{\mathbf{k+l}}(1+n^\pi_{\mathbf{l}})n^\pi_{\mathbf{k}}\}\cos(\omega^\sigma_{\mathbf{l+k}}-\omega^\pi_{\mathbf{l}}+\omega^\pi_{\mathbf{k}})(s-t)&\\ +\{n^\sigma_{\mathbf{k+l}}(1+n^\pi_{\mathbf{k}})n^\pi_{\mathbf{l}} - (1+n^\sigma_{\mathbf{k+l}})(1+n^\pi_{\mathbf{l}})n^\pi_{\mathbf{k}}\}\cos(\omega^\sigma_{\mathbf{l+k}}+\omega^\pi_{\mathbf{l}}-\omega^\pi_{\mathbf{k}})(s-t)],&\end{aligned} \tag{73}$$

Each terms in the brace on the r.h.s. of the equation can be interpreted as the gain minus loss processes as in the case of the σ transport equation. The time-evolution of the σ and π distribution functions are obtained by solving this coupled equations.

IV. SUMMARY AND CONCLUSIONS

We have developed a formalism to carry out coarse-grainings by using a time-dependent projection operator. As a result, we have obtained the Langevin-type equation without the time-convolution integral term, which is called the time-convolutionless equation. In this formalism, the irrelevant subsystem can have some time-dependence, and therefore, we can treat a more general nonequilibrium process where the mass, the temperature and so on are time-dependent. When we ignore the time-dependence of the projection operator, we can reproduce the result in the usual projection operator method [3]. Furthermore, the third term on the r.h.s. of the Eq.(29) represents the effect of the initial correlation. We thus interpret it as the fluctuation force even in the time-dependent projection operator method.

We have applied this formalism to the quantum field theoretical model which consists of the σ and π bosons. We have then obtained the coupled transport equations of the two bosons. Deriving the coupled transport equations, the correlation functions are calculated with respect to the local equilibrium density matrix. In other words, we have assumed that the time-evolution of the system can be well-approximated by the local equilibrium distribution function. The derived equations have the terms which are not seen in the usual Boltzmann equation. It is worth to study the effect of such terms on the transport equation, but we have dropped them in this paper. Each transport equation has the form which can be interpreted as the gain minus loss processes, which is the structure usually seen in the Boltzmann equation. Therefore, we can expect that the time-dependent projection operator method developed in this paper is a valid formalism to describe the nonequilibrium process.

In this paper, we have ignored the time-dependence of the nonperturbative Hamiltonian. In the general nonequilibrium process, it is possible to consider the situation where the mass of a particle changes with time. Then, the time-dependence of the nonperturbative Hamiltonian is caused by the time-dependent mass. This effect will be discussed in other paper.

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APPENDIX A: DEFINITION OF OPERATORS $\mathcal{C}$ AND $\mathcal{D}$

The definitions of the operators $\mathcal{C}(t,t_0)$ and $\mathcal{D}(t,t_0)$ are

$$\mathcal{C}(t,t_0) = U_0(t,t_0)e^{-iL(t-t_0)}, \tag{A1}$$
$$\mathcal{D}(t,t_0) = e_{\rightarrow}^{i\int_{t_0}^{t} dsLQ(s)}(U_0^Q)^{-1}(t,t_0), \tag{A2}$$

where

$$U_0(t,t_0) = e_{\rightarrow}^{i\int_{t_0}^{t} dsL_0(s,t_0)}, \tag{A3}$$
$$U_0^Q(t,t_0) = e_{\rightarrow}^{i\int_{t_0}^{t} dsL_0(s,t_0)Q(s)}. \tag{A4}$$

These operators are subject to the following differential equations:

$$\begin{aligned}\frac{d}{dt}\mathcal{C}(t,t_0) &= U_0(t,t_0)(iL_0(t,t_0) - iL)e^{-iL(t-t_0)} \\ &= -i\breve{L}_I(t,t_0)\mathcal{C}(t,t_0), \qquad (A5)\\ \frac{d}{dt}\mathcal{D}(t,t_0) &= e_{\rightarrow}^{i\int_{t_0}^{t} dsLQ(s)}(iL - iL_0(t,t_0))Q(t)(U_0^Q)^{-1}(t,t_0) \\ &= \mathcal{D}(t,t_0)i\breve{L}_I^Q(t,t_0), \qquad (A6)\end{aligned}$$

where

$$\breve{L}_I(t,t_0) = U_0(t,t_0)L_I(t,t_0)U_0^{-1}(t,t_0), \tag{A7}$$
$$\breve{L}_I^Q(t,t_0) = U_0^Q(t,t_0)L_I(t,t_0)Q(U_0^Q)^{-1}(t,t_0). \tag{A8}$$

The solutions of the above differential equations are

$$\mathcal{C}(t,t_0) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \breve{L}_I(t_1,t_0)\breve{L}_I(t_2,t_0)\cdots\breve{L}_I(t_n,t_0), \tag{A9}$$

$$\mathcal{D}(t,t_0) = 1 + \sum_{n=1}^{\infty} i^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \breve{L}_I^Q(t_1,t_0)\breve{L}_I^Q(t_2,t_0)\cdots\breve{L}_I^Q(t_n,t_0). \tag{A10}$$

APPENDIX B: TRANSFORMATION OF OPERATOR $\Sigma(T,T_0)$

The operator $\Sigma(t,t_0)$ can be rewritten as

$$\begin{aligned}
\Sigma(t,t_0) &= \int_{t_0}^{t} ds e^{-iL(t-s)}\{\dot{Q}(s) + P(s)iLQ(s)\} e_{\rightarrow}^{i\int_s^t d\tau LQ(\tau)} \\
&= \int_{t_0}^{t} ds e^{-iL(t-s)} \frac{d}{ds}(Q(s) e_{\rightarrow}^{i\int_s^t d\tau LQ(\tau)}) - \int_{t_0}^{t} ds e^{-iL(t-s)} \frac{d}{ds} e_{\rightarrow}^{i\int_s^t d\tau LQ(\tau)} \\
&= -P(t) + e^{-iL(t-t_0)} P(t_0) e_{\rightarrow}^{i\int_{t_0}^t d\tau LQ(\tau)} + \int_{t_0}^{t} ds e^{-iL(t-s)} iLP(s) e_{\rightarrow}^{i\int_s^t d\tau LQ(\tau)} \\
&= -P(t) + e^{-iL(t-t_0)} P(t_0) e_{\rightarrow}^{i\int_{t_0}^t d\tau LQ(\tau)} + \int_{t_0}^{t} ds \frac{d}{ds} e^{-iL(t-s)} e_{\rightarrow}^{i\int_s^t d\tau LQ(\tau)} \\
&= Q(t) - e^{-iL(t-t_0)} Q(t_0) e_{\rightarrow}^{i\int_{t_0}^t d\tau LQ(\tau)} \\
&= Q(t) - U_0^{-1}(t,t_0)\mathcal{C}(t,t_0)Q(t_0)\mathcal{D}(t,t_0) e_{\rightarrow}^{i\int_{t_0}^t ds L_0(s,t_0)Q(s)}.
\end{aligned} \tag{B1}$$

Using the mathematical induction, we confirm the following relation:

$$\begin{aligned}
&P(t)\Sigma(t,t_0)(Q(t)\Sigma(t,t_0))^n \\
&= [(-1)^{n-1}P(t)\{\tilde{Q}(t)(\tilde{\mathcal{C}}(t)-1)\}^n\tilde{Q}(t) + (-1)^{n-1}P(t)\{(\tilde{\mathcal{C}}(t)-1)\tilde{Q}(t)\}^{n+1}] \\
&\quad + P(t)\sum_{l=0}^{n-1}(-1)^l\{\tilde{Q}(t)(\tilde{\mathcal{C}}(t)-1)\}^l\tilde{Q}(t)(\tilde{\mathcal{D}}(t)-1)(\tilde{Q}(t)\Sigma(t,t_0))^{n-1-l} \\
&\quad + P(t)\sum_{l=0}^{n-1}(-1)^l\{(\tilde{\mathcal{C}}(t)-1)\tilde{Q}(t)\}^{l+1}(\tilde{\mathcal{D}}(t)-1)(\tilde{Q}(t)\Sigma(t,t_0))^{n-1-l} \\
&\quad - P(t)\sum_{l=0}^{n}(-1)^l\{\tilde{Q}(t)(\tilde{\mathcal{C}}(t)-1)\}^l\tilde{Q}(t)(\tilde{\mathcal{D}}(t)-1)(\tilde{Q}(t)\Sigma(t,t_0))^{n-l} \\
&\quad - P(t)\sum_{l=0}^{n}(-1)^l\{(\tilde{\mathcal{C}}(t)-1)\tilde{Q}(t)\}^{l+1}(\tilde{\mathcal{D}}(t)-1)(\tilde{Q}(t)\Sigma(t,t_0))^{n-l},
\end{aligned} \tag{B2}$$

where

$$\tilde{Q}(t) = U_0^{-1}(t,t_0)Q(t_0)U_0^Q(t,t_0), \tag{B3}$$
$$\tilde{\mathcal{C}}(t) = U_0^{-1}(t,t_0)\mathcal{C}(t,t_0)U_0(t,t_0), \tag{B4}$$
$$\tilde{\mathcal{D}}(t) = (U_0^Q)^{-1}(t,t_0)\mathcal{D}(t,t_0)U_0^Q(t,t_0). \tag{B5}$$

Here, n is an integer and $n \geq 1$. In this derivation, we use the following relation:

$$Q(t)\tilde{Q}(t) = Q(t), \tag{B6}$$

which can be proved from the condition (19). The second and third terms in $P(t)\Sigma(t,t_0)(Q(t)\Sigma(t,t_0))^n$ and the fourth and fifth terms in $P(t)\Sigma(t,t_0)(Q(t)\Sigma(t,t_0))^{n-1}$ cancel. The fourth and fifth terms in $P(t)\Sigma(t,t_0)(Q(t)\Sigma(t,t_0))^n$ and the second and third terms in $P(t)\Sigma(t,t_0)(Q(t)\Sigma(t,t_0))^{n+1}$ also cancel. Therefore, only the first term survives. As a result, all the terms including $\tilde{\mathcal{D}}(t)$ disappear.

Note that the relation $\Sigma(t,t_0) = \Sigma(t,t_0)Q(t)$ which is derived by Eq. (16). We thus find

$$\begin{aligned}
P(t)\Sigma(t,t_0)\frac{1}{1-\Sigma(t,t_0)} &= P(t)\Sigma(t,t_0)\frac{1}{1-Q(t)\Sigma(t,t_0)} \\
&= P(t)\Sigma(t,t_0)\sum_{n=0}^{\infty}(Q(t)\Sigma(t,t_0))^n \\
&= -P(t)\sum_{n=0}^{\infty}[\{-\tilde{Q}(t)(\tilde{\mathcal{C}}(t)-1)\}^n\tilde{Q}(t) - \{-(\tilde{\mathcal{C}}(t,t_0)-1)\tilde{Q}(t)\}^{n+1}] \\
&= -P(t)U_0^{-1}(t,t_0)\sum_{n=0}^{\infty}[\{-Q(t)(\mathcal{C}(t,t_0)-1)\}^n Q(t) \\
&\quad -\{-(\mathcal{C}(t,t_0)-1)Q(t)\}^{n+1}]U_0(t,t_0) \\
&= -P(t)U_0^{-1}(t,t_0)Q(t)\frac{1}{1+(\mathcal{C}(t,t_0)-1)Q(t)}U_0(t,t_0) \\
&\quad -P(t)U_0^{-1}(t,t_0)(\mathcal{C}(t,t_0)-1)Q(t)\frac{1}{1+(\mathcal{C}(t,t_0)-1)Q(t)}U_0(t,t_0) \\
&= -P(t)U_0^{-1}(t,t_0)\mathcal{C}(t,t_0)Q(t)\frac{1}{1+(\mathcal{C}(t,t_0)-1)Q(t)}U_0(t,t_0).
\end{aligned} \tag{B7}$$

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